FTD-MT-67-19

AD 675377

# FOREIGN TECHNOLOGY DIVISION

## SPECTRAL TRANSPARENCY OF ALMOST MONODISPERSED SYSTEMS

by

K. S. Shifrin and A. Ya. Perel'man

OCT 1968

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35

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FTD-MT-67-19

# EDITED MACHINE TRANSLATION

SPECTRAL TRANSPARENCY OF ALMOST MONODISPERSED SYSTEMS

By: K. S. Shifrin and A. Ya. Perel'man

English pages: 32

SOURCE: Leningrad. Glavnaya Geofizicheskaya Observatoriya. Trudy. (Leningrad. Main Geophysical Observatory. Transactions), No. 170, 1965, pp. 37-60.

UR/2531-65-000-170 TT7501968



PREPARED BY:

TRANSLATION DIVISION
FOREIGN TECHNOLOGY DIVISION
WP-AFB, OHIO.

FTD-MT-67-19 Date 17 Nov 1967

AFLC-WPAFB-SEP 68 98

ITIS INDEX CONTROL FORM

| 01 Acc Nr | 68 Translation Nr | 65 X Ref Acc Nr | 76 Reel/Frame Nr |
|---|---|---|---|
| TT7501968 | MT6700019 | AT5025226 | 1881 2264 |

| 97 Header Clas | 63 Clas | 64 Control Markings | 94 Expansion | 40 Ctry Info |
|---|---|---|---|---|
| UNCL | UNCL, 0 | 0 | | UR |

| 02 Ctry | 03 Ref | 04 Yr | 05 Vol | 06 Iss | 07 B. Pg. | 45 E. Pg. | 10 Date |
|---|---|---|---|---|---|---|---|
| UR | 2531 | 65 | 000 | 170 | 0037 | 0060 | NONE |

Transliterated Title
SPEKTRAL'NAYA PROZRACHNOST' POCHTI MONODISPERSNYKH SISTEM

09 English Title
SPECTRAL TRANSPARENCY OF ALMOST MONODISPERSED SYSTEMS

43 Source
LENINGRAD. GLAVNAYA GEOFIZICHESKAYA OBSERVATORIYA. TRUDY (RUSSIAN)

| 42 Author | 98 Document Location |
|---|---|
| SHIFRIN, K. S. | |
| **16 Co-Author** | **47 Subject Codes** |
| PEREL'MAN, A. YA. | 20, 04 |
| **16 Co-Author** | **39 Topic Tags:** atmospheric optics, atmospheric transparency, particle distribution, light scattering, atmospheric scatter |
| NONE | |
| **16 Co-Author** | |
| NONE | |
| **16 Co-Author** | |
| NONE | |

ABSTRACT: The effect of the parameters of polydispersion, particularly the width of distribution $\Delta r$, on the transparency of the system is investigated. Monodispersed scattering of light is regarded as the limiting case of polydispersed scattering, which can be represented by a series of delta-shaped distribution curves whose properties are used to compute, by a modified saddle-point method, the integral representing the polydispersed scattering coefficient. Scattering in an almost monodispersed system is regarded as monodisperse with a correction factor. Analysis begins with consideration of a polydispersed system of particles whose optical properties differ little from those of the surrounding medium. Changes in the spectral transparency of the system are investigated along two lines In the first, different distribution widths are considered for constant mean-square radius $\bar{r}_2$ (transparency remains constant for small $\lambda$). In the second line of investigation, the mode of the distribution $r_M$ is fixed. Formulas are derived for determining transparency with constant $\bar{r}_2$ and $r_M$. It is assumed that the particle-size (radii) spectrum is described by a gamma-distribution and the scattering cross section is an arbitrary analytic function whose argument is proportional to the product $r\nu$ ($\nu$ is the wave number). An expression is derived for the optical characteristics (for example, transparency) of almost monodispersed systems. The range of applicability of these formulas is evaluated. Calculations are presented which illustrate details of the spectral structure of transparency as the distribution width is narrowed (transition to the monodispersed case). The connection between dimensionless characteristics of transparency for different linear scales is established, and a formula is derived for determining transparency when the linear scale is changed. Curves of the spectral transparency of different polydispersed systems are presented to illustrate the application of the formulas. Orig. art. has: 9 figures, 136 formulas, and 4 tables. English trans. 32 pages.

Form FTD Feb 67 0-90

U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

| Block | Italic | Transliteration | Block | Italic | Transliteration |
|---|---|---|---|---|---|
| А а | *А а* | A, a | Р р | *Р р* | R, r |
| Б б | *Б б* | B, b | С с | *С с* | S, s |
| В в | *В в* | V, v | Т т | *Т т* | T, t |
| Г г | *Г г* | G, g | У у | *У у* | U, u |
| Д д | *Д д* | D, d | Ф ф | *Ф ф* | F, f |
| Е е | *Е е* | Ye, ye; E, e* | Х х | *Х х* | Kh, kh |
| Ж ж | *Ж ж* | Zh, zh | Ц ц | *Ц ц* | Ts, ts |
| З з | *З з* | Z, z | Ч ч | *Ч ч* | Ch, ch |
| И и | *И и* | I, i | Ш ш | *Ш ш* | Sh, sh |
| Й й | *Й й* | Y, y | Щ щ | *Щ щ* | Shch, shch |
| К к | *К к* | K, k | Ъ ъ | *Ъ ъ* | " |
| Л л | *Л л* | L, l | Ы ы | *Ы ы* | Y, y |
| М м | *М м* | M, m | Ь ь | *Ь ь* | ' |
| Н н | *Н н* | N, n | Э э | *Э э* | E, e |
| О о | *О о* | O, o | Ю ю | *Ю ю* | Yu, yu |
| П п | *П п* | P, p | Я я | *Я я* | Ya, ya |

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

SPECTRAL TRANSPARENCY OF ALMOST MONODISPERSED SYSTEMS

K. S. Shifrin and A. Ya. Perel'man

A modified saddle-paint method for the obtaining of approximate formulas of transparency of almost monodispersed systems is used. As the model of delta-shaped sequence describing the distribution of radii of particles in such systems gammadistributions are accepted. There is estimated the domain of applicability of the obtained formulas. Calculations are given illustrating the appearance of components of the spectral structure of the transparency according to a decrease in width of the distribution (transition to the monodispersed case). A formula of the transition for transparency with a change in linear scale of the problem is derived.

## § 1. Introduction. Formulation of the Problem

Optical properties of real polydispersional systems depend both on characteristics of individual scattering and on characteristics of the microstructure-parameters of the distribution of particles according to dimensions. If the question on the influence of the first characteristics was repeatedly investigated and with respect to it there is extensive literature, then the question on the influence of distribution parameters on a pattern of light diffusion is known in general rather little [1, 2]. In this work we will examine the influence of parameters of polydisperison on transparency of the system.

The simplest polydispersional distribution is characterized by the following parameters: vertical scale expressed in terms of density of particles N and being simply the parameter of normalization; horizontal linear scale (average dimension $\bar{r}$ or modal $r_M$ or some other) and width of distribution $\Delta r$, defined by a certain method. The first two parameters are present in individual scattering. The width of the distribution of $\Delta r$ is parameter characteristic for polydispersional scattering. The purpose of this work is to investigate the influence

of this parameter $\Delta r$ on the spectral transparency. Although the qualitative side of question is clear (with a decrease in $\Delta r$ the pattern of light diffusion should approach that which follows the from theory of scattering for the separate particle), the quantitative side requires detailed investigation.

It is interesting to clarify with what $\Delta r$ appear certain interference components of the individual pattern of scattering and in particular, with what $\Delta r$ on the curve of spectral transmittance is it possible to observe the replacement Rayleigh and anti-Rayleigh segments of the (wave on the curve of transparency) etc.

With investigation of the transition from polydispersional scattering to monodispersion considerable calculating and analytic difficulties appear in the limiting region of small $\Delta r$, i.e., in the case of almost monodispersed systems. This is connected with the fact that then the distribution curve will cease to be "ordinary" function and degenerates into the generalized function, the – $\delta$-Dirac function. The general method of examining almost monodispersed systems was developed in [3]. As the basis of it lies the idea according to which monodispersed scattering can be considered as the limit of polydispersional sequence delta-shaped distribution curves. Using the property of functions of the delta-shaped sequence, we will apply to the calculation of the integral representing polydispersional coefficient of scattering the somewhat modified saddle-paint method. It will be shown that scattering in an almost monodispersed system can be to considered, as monodispersed with a certain correction. For calculation of the correction there is obtained a very simple formula. The less $\Delta r$, the proposed method is found to be more effective since it by its nature is adjusted to almost monodispersed systems.

Let us examine the polydispersional system of particles whose optical properties do not very greatly deviate from properties of the environment. For the scattering coefficient of light on the separate particle $k(r, \lambda)$ in this case we have

$$k(r, \lambda) = 2\pi r^2 K(\delta), \tag{1.1}$$

where

$$K(\delta) = 1 - \frac{\sin 2\delta}{\delta} + \frac{1 - \cos 2\delta}{2\delta^2}, \tag{1.2}$$

$$\delta = 2\pi(m-1)\frac{r}{\lambda} = \beta r \nu^*, \quad \nu^* = \frac{1}{\lambda}, \quad \beta = 2\pi(m-1). \tag{1.3}$$

Here r is the radius of the particles, $\lambda$ – wavelength, m – index of refraction, and $\nu^*$ – wave number.

The polydispersional coefficient of scattering (transparency) $g^*(\nu^*)$ for the wave number $\nu^*$ will be

$$g^*(\nu^*) = \int_0^\infty 2\pi r^2 K(\delta) f^*(r)\, dr, \tag{1.4}$$

where $f^*(r)$ is the distribution function of particles by dimensions. Below we will take it for gammadistribution of the form

$$f^*_\mu(r) = A r^\mu e^{-br}. \tag{1.5}$$

Properties of this distribution were studies in [4]. There, in particular, it was shown that the standardized (with respect to the quantity of particles N per unit of volume) distribution (1.5) will be

$$f^*_\mu(r) = N \frac{b^{\mu+1}}{\Gamma(\mu+1)} r^\mu e^{-br} \tag{1.6}$$

and its relative width $\Delta\varepsilon$ is determined by the formula

$$\Delta\varepsilon = \frac{\Delta r}{r_M} = \frac{2.48}{\sqrt{\mu}}. \tag{1.7}$$

Here $\Delta r$ is the absolute width of the curve on the level $\frac{1}{2} f^*_\mu(r_M)$, and $r_M$ is the mode of distribution. Our problem now is reduced to the calculation of the integral (1.4) and to the investigation of changes in curve $g_\mu^*(\nu^*)$ depending upon $\Delta\varepsilon$ for the corresponding distribution $f_\mu^*(r)$. This calculation will be necessary to make for very large $\mu$, in order to estimate the limit at $\mu \to \infty$. Let us call that at small $\mu$ integers simple formulas for $g_\mu^*(\nu^*)$ were shown earlier in [5].

The change in spectral transmittance of the system is investigated with two schemes of the experiment. In first scheme there are considered distributions of different width at a fixed value of the mean-square radius $\bar{r}_2$, and in the second, at a fixed value of the mode of distribution $r_M$.

Fixing of the mean-square radius possesses great convenience in examining different schemes of experiments with light scattering. In this case with transition into the region of geometric optics ($\lambda \to 0$) all models have common asymptote. For instance, for spectral transmittance of polydispersional systems we have

$$g^*(\nu^*) = \int_0^\infty 2\pi r^2 K(\delta) f^*(r)\, dr \underset{\text{when } \lambda \to 0}{\longrightarrow} \int_0^\infty 2\pi r^2 f^*(r)\, dr = 2\pi \bar{r}_2^2 N. \tag{1.8}$$

Consequently for any models with identical $\bar{r}_2$ the transparency in the region of small $\lambda$ will be identical. Formulas for transparency with fixed $\bar{r}_2$ are derived in § 2-5.

In certain cases for the basic characteristic of distribution it is convenient to take not $\bar{r}_2$ but the modal value $r_M$. This quantity is more graphic than $\bar{r}_2$, represents the form of the distribution curve and is the most convenient linear characteristic in the experiment. Formulas for transparency with fixed $r_M$ are given in § 6.

All transformations mentioned below are made over dimensionless variables. Let us assume that as unity of scale there is accepted radius $r_0$. Then the bond between the appropriate measured and dimensionless values yields the formulas:

$$r = r_0 a, \quad f_\mu^*(r) = \frac{f_\mu(a)}{r_0^4}, \quad \frac{x}{2} = y = \beta \nu^* r_0, \quad g_\mu(y) = r_0 g_\mu^*(\nu^*). \tag{1.9}$$

Here r and a, $f_\mu^*(r)$ and $f_\mu(a)$, $g_\mu^*(\nu^*)$ and $g_\mu(y)$ are respectively the measured and dimensionless radii, and functions of gammadistributions of particles by radii, and polydispersional scattering coefficients (transparency) corresponding to these distributions; x and y are dimensionless analogies of the wave number $\nu^*$; $\beta$ is determined by formula (1.3). On the basis (1.6) and (1.9) we obtain

$$f_\mu(a) = N r_0^3 \frac{\Delta^{\mu+1}}{\Gamma(\mu+1)} a^\mu e^{-\Delta a}, \quad \Delta = \beta r_0. \tag{1.10}$$

The dimensionless parameter $\Delta$ is a function of $\mu$ the form of which is determined by the selection of the linear scale $r_0$. Thus if $r_0 = r_M$ ($r_M$ is the mode of distribution), then $\Delta = \mu$; if $r_0 = \bar{r}$ ($\bar{r}$ is the mean radius of distribution), then $\Delta = \mu + 1$; if $r_0 = \bar{r}_2$ ($\bar{r}_2$ is the mean-square radius), then $\Delta = \sqrt{(\mu+2)(\mu+1)}$ etc. The integral (1.4) in dimensionless variables has the form

$$g_\mu\left(\frac{x}{2}\right) = \int_0^\infty K\left(\frac{x}{2}, a\right) m_\mu(a)\, da, \quad m_\mu(a) = 2\pi a^2 f_\mu(a). \tag{1.11}$$

According to (1.10) and (1.11) we find

$$m_\mu(a) = \gamma_\mu a^{\mu+2} e^{-\Delta a}, \quad \gamma_\mu = \frac{\Delta^{\mu+1}}{\Gamma(\mu+1)} \gamma_0, \quad \gamma_0 = 2\pi N r_0^3. \tag{1.12}$$

## § 2. Transparency of a System with Gammadistribution

Let us derive a formula for transparency of a system corresponding to distribution (1.10) at the arbitrary linear scale of $r_0$. In virtue of (1.2) (1.11) and (1.12) we have

$$g_\mu\left(\frac{x}{2}\right)=\gamma_\mu\int_0^\infty a^{\mu+2}e^{-\Delta a}\left(1-2\frac{\sin ax}{ax}+2\frac{1-\cos ax}{a^2x^2}\right)da=$$
$$=\gamma_\mu\left[\frac{\Gamma(\mu+3)}{\Delta^{\mu+3}}-\frac{2}{x}J_1+\frac{2}{x^2}\frac{\Gamma(\mu+1)}{\Delta^{\mu+1}}-\frac{2}{x^2}J_2\right], \qquad (2.1)$$

where

$$J_1=\int_0^\infty a^{\mu+1}e^{-\Delta a}\sin ax\,da,\quad J_2=\int_0^\infty a^{\mu}e^{-\Delta a}\cos ax\,da. \qquad (2.2)$$

Using Euler the formula, we find

$$J_1=\frac{1}{2i}\left\{\frac{\Gamma(\mu+2)}{(\Delta-xi)^{\mu+2}}-\frac{\Gamma(\mu+2)}{(\Delta+xi)^{\mu+2}}\right\},$$
$$J_2=\frac{1}{2}\left\{\frac{\Gamma(\mu+1)}{(\Delta-xi)^{\mu+1}}+\frac{\Gamma(\mu+1)}{(\Delta+xi)^{\mu+1}}\right\}.$$

Hence

$$J_1=\frac{\Gamma(\mu+2)}{\Delta^{\mu+2}}\cdot\frac{\operatorname{Im}\left(1+\frac{x}{\Delta}i\right)^{\mu+2}}{\left(1+\frac{x^2}{\Delta^2}\right)^{\mu+2}},\quad J_2=\frac{\Gamma(\mu+1)}{\Delta^{\mu+1}}\cdot\frac{\operatorname{Re}\left(1+\frac{x}{\Delta}i\right)^{\mu+1}}{\left(1+\frac{x^2}{\Delta^2}\right)^{\mu+1}}. \qquad (2.3)$$

On the basis of formulas (2.3), (1.12) and identity

$$\frac{\left(1+\frac{x}{\Delta}i\right)^k}{\left(1+\frac{x^2}{\Delta^2}\right)^{\frac{k}{2}}}=\cos^k\omega\,(\cos k\omega+i\sin k\omega) \qquad (2.4)$$

the expression (2.1) takes the form

$$g_0\left(\frac{x}{2}\right)=\gamma_0\left\{\frac{(\mu+2)(\mu+1)}{\Delta^2}+\frac{2}{x^2}-\frac{2\cos^{\mu+1}\omega}{x}\left[\frac{\mu+1}{\Delta}\cos\omega\sin((\mu+2)\omega)+\right.\right.$$
$$\left.\left.+\frac{\cos((\mu+1)\omega)}{x}\right]\right\}. \qquad (2.5)$$

Here

$$\gamma_0=2\pi Nr_0^2,\quad \operatorname{tg}\omega=\frac{x}{\Delta}. \qquad (2.6)$$

Further (up to § 6) as a linear scale we take the mean-square radius $\overline{r}_2$, and correspondingly in formulas (1.9), (1.10), (1.12), (2.5) and (2.6) one should assume $r_0 = \overline{r}_2$ and $\Delta = \varepsilon$, and

$$\varepsilon = \sqrt{(\mu+2)(\mu+1)}. \tag{2.7}$$

Consequently, for the distribution

$$m_\mu(a) = \tau \frac{\varepsilon^{\mu+1}}{\Gamma(\mu+1)} a^{\mu+2} e^{-\varepsilon a} \tag{2.8}$$

transparency is determined by the expression

$$g_\mu\left(\frac{x}{2}\right) = \tau\left\{1 + \frac{2}{x^2} - \frac{2\cos^{\mu+1}\omega}{x}\left[\sqrt{\frac{\mu+1}{\mu+2}}\cos\omega\sin((\mu+2)\omega) + \right.\right.$$
$$\left.\left. + \frac{\cos((\mu+1)\omega)}{x}\right]\right\}, \tag{2.9}$$

where

$$\tau = 2\pi N\overline{r}_2^2, \quad \operatorname{tg}\omega = \frac{x}{\sqrt{(\mu+2)(\mu+1)}}. \tag{2.10}$$

§ 3. Transparency of Almost Monodispersed Systems

In § 3 and 4 it is assumed that the spectrum of particles-size of radii is described by gammadistribution (2.8) and the diameter of scattering $2K(\delta)$ is assigned by arbitrary analytic function whose argument is proportional to the product $r\nu$* [and, according to (1.9), the product $ay$][1]. § 3 formally derives the expression for optical characteristics (for instance, transparency) of almost monodispersed systems ($\mu \gg 1$), and in § 4 the foundation for this derivation is given. In § 5 in detail there is considered the case when the diameter of scattering $2K(\delta)$ is defined by expression (1.2).

It is not difficult to check that $m_\mu(a)$ from (1.11) is the $\delta$-shaped sequence with respect to $a = 1$ (when $\mu \to \infty$). Therefore, in carrying out integration in (1.11) it is natural to use the idea of the saddle-paint method.

Let us expand $K(ay)$ in Taylor series near point $a = 1$. We have

$$K(ay) = \sum_{n=0}^{\infty} \frac{K^{(n)}(y)\,y^n}{n!}(a-1)^n. \tag{3.1}$$

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[1]Let us note that $\delta = ay$ if a and y are assigned by formula (1.9) and $\delta$ by formula (1.3).

Substituting this expression into formula (1.11), considering (2.8), (2.10) and the relation 2y = x (1.9) we obtain

$$g_\mu(y) = \gamma \frac{\tau^{\mu+1}}{\Gamma(\mu+1)} \sum_{n=0}^{\infty} \frac{y^n K^{(n)}(y)}{n!} \int_0^{\infty} a^{\mu+2} e^{-\tau a} (a-1)^n da. \tag{3.2}$$

Using the formula Dirichlet for sums

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} c_{nm} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} c_{nm} \tag{3.3}$$

and equality

$$(a-1)^n = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} a^m, \quad \int_0^{\infty} a^{\mu+m+2} e^{-\tau a} da = \frac{\Gamma(\mu+m+3)}{\tau^{\mu+m+3}}, \tag{3.4}$$

we find

$$g_\mu(y) = \gamma \sum_{m=0}^{\infty} \frac{(\mu+m+2)(\mu+m+1)\ldots(m+1)}{\tau^{m+2}} \times$$

$$\times \sum_{n=m}^{\infty} \binom{n}{m} (-1)^{n-m} \frac{y^n K^{(n)}(y)}{n!}. \tag{3.5}$$

It is easy to show (see § 4 p. 1) that the internal sum in (3.5) will be

$$\sum_{n=m}^{\infty} \binom{n}{m} (-1)^{n-m} \frac{y^n K^{(n)}(y)}{n!} = \frac{y^m K^{(m)}(0)}{m!}. \tag{3.6}$$

Let us designate

$$\beta(\mu, m) = \frac{\prod_{k=1}^{m+2} \left(1 + \frac{k}{\mu}\right)}{\left[\left(1 + \frac{1}{\mu}\right)\left(1 + \frac{2}{\mu}\right)\right]^{\frac{m+2}{2}}}. \tag{3.7}$$

$$a_m = \frac{K^{(m)}(0)}{m!}. \tag{3.8}$$

According to (3.5)-(3.8),

$$g_\mu(y) = \gamma \sum_{m=0}^{\infty} \beta(\mu, m) a_m y^m, \tag{3.9}$$

where, obviously,

$$K(y) = \sum_{m=0}^{\infty} a_m y^m. \tag{3.10}$$

Expansion (3.9) will be called as basic series.

From (3.10) it follows what

$$\begin{aligned} &\sum_{m=0}^{\infty} (c_0 m^n + c_1 m^{n-1} + \ldots + c_n) a_m y^m = \\ &= c_0 K_{(n)}(y) + c_1 K_{(n-1)}(y) + \ldots + c_n K(y), \end{aligned} \tag{3.11}$$

where

$$K_{(l)}(y) = y\left(y \ldots (yK'(y))' \ldots\right)' \quad (l \text{ primes}). \tag{3.12}$$

Let us expand $\beta(\mu, m)$ in series with respect to $\frac{1}{\mu}$. After simple computations we obtain

$$\beta(\mu, m) = 1 + \frac{m^2 + 2m}{2} \cdot \frac{1}{\mu} + \frac{3m^4 + 8m^3 - 18m^2 - 44m}{24} \cdot \frac{1}{\mu^2} + 0\left(\frac{1}{\mu^3}\right). \tag{3.13}$$

This means that expression (3.9) on the basis (3.11) and (3.13) can be recorded in the form

$$\begin{aligned} g_\mu(y) = \gamma \Big\{ K(y) + \frac{K_{(2)}(y) + 2K_{(1)}(y)}{24} \cdot \frac{1}{\mu} + \\ + \frac{3K_{(4)}(y) + 8K_{(3)}(y) - 18K_{(2)}(y) - 44K_{(1)}(y)}{2} \cdot \frac{1}{\mu^2} + 0\left(\frac{1}{\mu^3}\right) \Big\}, \end{aligned} \tag{3.14}$$

where $K_n(y)$ is calculated with respect to $K(y)$ with the help of formula (3.12) and $\gamma$ is givenly (2.10).

## § 4. Derivation of Certain Auxiliary Equalities. Investigation of the Convergence of the Basic Series

1. Let us prove the accuracy of identity (3.6). We have

$$\sum_{n=m}^{\infty}\binom{n}{m}(-1)^{n-m}\frac{y^n K^{(n)}(y)}{n!}=\frac{1}{m!}\sum_{n=m}^{\infty}(-1)^{n-m}\frac{y^n K^{(n)}(y)}{(n-m)!}=$$

$$=\frac{y^m}{m!}\sum_{n=m}^{\infty}\frac{(-1)^{n-m}y^{n-m}K_m^{(n-m)}(y)}{(n-m)!}=\frac{y^m}{m!}\sum_{r=0}^{\infty}\frac{(-1)^r y^r K_m^{(r)}(y)}{r!}. \qquad (4.1)$$

Here there was introduced the function

$$K_m(y)=K^{(m)}(y),$$

so that

$$K^{(n)}(y)=K_m^{(n-m)}(y),$$

and then it is assumed that $n - m = r$. Let us expand now $K_m(z)$ in Taylor series at point y

$$K_m(z)=\sum_{n=0}^{\infty}\frac{K_m^{(n)}(y)(z-y)^n}{n!}.$$

Relying here $z = 0$, we find

$$\sum_{n=0}^{\infty}\frac{K_m^{(n)}(y)(-1)^n y^n}{n!}=K_m(0)=K^{(m)}(0). \qquad (4.2)$$

Substituting (4.2) into the last equality, (4.1), we will obtain (3.6).

2. Formula (3.11) permits finding the sums

$$\phi_n(y)=\sum_{m=0}^{\infty}P_n(m)a_m y^m. \qquad (4.3)$$

where $P_n(m)$ is a polynomial of the n power

$$P_n(m)=c_0 m^n+c_1 m^{n-1}+\ldots+c_n. \qquad (4.4)$$

in terms of derivatives K(y) the expansion of which in Taylor series has the form

$$K(y) = \sum_{m=0}^{\infty} a_m y^m, \quad a_m = \frac{K^{(m)}(0)}{m!}. \tag{4.5}$$

However with this, according to (3.11) and (3.12), it is necessary to start from direct differentiation of function K(y), which is sometimes inconvenient. In particular, if K(y) contains in the denominator $y^t$ (t is any), then it is more preferable to begin from the differentiation $y^t K(y)$. Let us show one of the methods of calculation $\psi_n(y)$ with initial differentiation of $y^t K(y)$.

Let us assume that, for instance, t = 2. Let us designate

$$\frac{1}{y}(y^2 K(y))' = \tilde{K}(y), \quad (y^2 K(y))'' = K^*(y). \tag{4.6}$$

Obviously,

$$K^*(y) = \sum_{m=0}^{\infty} a_m^* y^m, \quad a_m^* = (m+2)(m+1)a_m. \tag{4.7}$$

Using the Horner method, we consecutively find

$$P_n(m) = (m+2)P_{n-1}(m) + b_n = (m+2)\left[(m+1)P_{n-2}(m) + b_{n-1}\right] + b_n,$$

i.e.,

$$P_n(m) = (m+2)(m+1)P_{n-2}(m) + b_{n-1}(m+2) + b_n, \tag{4.8}$$

where

$$P_{n-2}(m) = b_0 m^{n-2} + b_1 m^{n-3} + \ldots + b_{n-2}. \tag{4.9}$$

Substituting (4.8) and (4.3), we find

$$\Psi_n(y) = \Psi^*_{n-2}(y) + b_{n-1}\tilde{K}(y) + b_n K(y), \tag{4.10}$$

where

$$\Psi^*_{n-2}(y) = \sum_{m=0}^{\infty} P_{n-2}(m) a_m^* y^m. \tag{4.11}$$

Thus the calculation (4.3) with respect to (4.5) leads with the help of expansion of (4.8) to the calculation (4.11) with respect to (4.7), in other words the power of the polynomial ($P_n(m)$ in (4.3) is lowered by t units (t = 2). Analogously, we can continue the lowering of the power of the polynomial ($P_{n-2}(m)$ by any quantity of t units. But practically, for calculation of $\psi^*_{n-2}(y)$, it is more convenient to apply the formula (3.11) in which K(y) is replaced by $K^*(y)$,

$$\Psi^*_{n-2}(y) = b_0 K^*_{(n-2)}(y) + b_1 K^*_{(n-3)}(y) + \\ + \dots + b_{n-3} K^*_{(1)}(y) + b_{n-2} K^*(y). \tag{4.12}$$

Here, obviously,

$$K^*_l(y) = y\left(y \dots (yK^{*\prime}(y))' \dots\right)', \ (l \text{ primes}). \tag{4.13}$$

Consequently, for (4.3) and (4.4) there has place

$$\Psi_n(y) = \sum_{l=1}^{n-2} b_{n-2-l} K^*_{(l)}(y) + b_{n-2} K^*(y) + \\ + b_{n-1} \tilde{K}(y) + b_n K(y). \tag{4.14}$$

where

$$P_n(m) = (m+2)(m+1) \sum_{l=0}^{n-2} b_{n-2-l} m^l + b_{n-1}(m+2) + b_n. \tag{4.15}$$

Let us call that this expansion is easily obtained by the method of Horner. Formulas (4.14) and (4.15), allowing the summing of series (4.3), will be used in § 5 and 6.

3. We investigate now the basic series (3.9) on the convergence. The expansion of (3.9) constitutes power series with respect to y. The radius of convergence R of this series is determined by the Cauchy-Adamar formula

$$R^{-1} = \overline{\lim_{m\to\infty}} |\beta(\mu, m) a_m|^{\frac{1}{m}}, \quad a_m = \frac{K^{(m)}(0)}{m!}. \tag{4.16}$$

Using formula (3.7) let us estimate $\frac{\beta(\mu, m)}{m!}$. We have

$$\frac{\beta(p,m)}{m!} < \frac{\left(1+\frac{m+2}{p}\right)\left(1+\frac{m+1}{p}\right)}{\left[\left(1+\frac{2}{p}\right)\left(1+\frac{1}{p}\right)\right]^{\frac{m+1}{2}}}\left[\frac{1}{p}+\frac{\ln m + C + \varepsilon(m)}{m}\right]^m. \qquad (4.17)$$

where C = 0.577 is the Euler constant and $\varepsilon(m) \to 0$ at $m \to \infty$. With the derivation of (4.17) there was applied the inequality of Cauchy, and the asymptotic formula for partial sums of harmonic series was used namely:

$$\prod_{k=1}^{m}\left(\frac{1}{p}+\frac{1}{k}\right) < \left\{\frac{\sum_{k=1}^{m}\left(\frac{1}{p}+\frac{1}{k}\right)}{m}\right\}^m = \left\{\frac{1}{p}+\frac{1}{m}\left[\ln m + C + \varepsilon(m)\right]\right\}^m.$$

This means

$$R^{-1} < \lim_{m\to\infty}\left[\frac{\left(1+\frac{m+2}{p}\right)\left(1+\frac{m+1}{p}\right)}{\left(1+\frac{2}{p}\right)\left(1+\frac{1}{p}\right)}\right]^{\frac{1}{m}} \times$$

$$\times \lim_{m\to\infty}\frac{\frac{1}{p}+\frac{\ln m + C + \varepsilon(m)}{m}}{\left[\left(1+\frac{2}{p}\right)\left(1+\frac{1}{p}\right)\right]^{\frac{1}{2}}}\ \overline{\lim_{m\to\infty}}\,|K^{(m)}(0)|^{\frac{1}{m}}$$

and since

$$\lim_{m\to\infty}(a+bm)^{\frac{1}{m}} = 1, \qquad (4.18)$$

we finally obtain [see (2.7)]

$$R > \frac{p\sqrt{\left(1+\frac{2}{p}\right)\left(1+\frac{1}{p}\right)}}{\overline{\lim\limits_{m\to\infty}}\sqrt[m]{|K^{(m)}(0)|}} = \frac{\sqrt{(p+2)(p+1)}}{\overline{\lim\limits_{m\to\infty}}\sqrt[m]{|K^{(m)}(0)|}}. \qquad (4.19)$$

Consequently the series (3.9) absolutely converges, in any case at

$$|y| < \frac{\sqrt{(p+2)(p+1)}}{\overline{\lim\limits_{m\to\infty}}\sqrt[m]{|K^{(m)}(0)|}}. \qquad (4.20)$$

4. In the absolutely converging series it is possible to transpose the members by any method. Let us transpose members of the series (3.9) having disposed it by powers $\frac{1}{\mu}$. Again the obtained series

$$g_{\mu}(y) = \tau \sum_{n=0}^{\infty} \frac{\xi(n, y)}{\mu^{n}} \tag{4.21}$$

knowingly converges with y and $\mu$, of connected (4.20) and (2.7).

Let us determine the form of coefficients of expansion (4.21) [first three members of this expansion are given in (3.14]. We preliminarily will show that for $\beta(\mu, m)$ from (3.7) takes place the formula ($\mu > 1$)

$$\beta(\mu, m) = \sum_{n=0}^{\infty} \frac{P_{2n}(m)}{\mu^{n}}, \tag{4.22}$$

where $P_{2n}(m)$ is the polynomial whose power does not exceed 2n [first three members of the series (4.22) are given in formula (3.13)].

Really

$$\beta(\mu, m) = \left[\left(1+\frac{2}{\mu}\right)\left(1+\frac{1}{\mu}\right)\right]^{-\frac{m+2}{2}} \alpha(\mu, m), \tag{4.23}$$

where

$$\alpha(\mu, m) = \prod_{k=1}^{m+2}\left(1+\frac{k}{\mu}\right). \tag{4.24}$$

Calculating the product of series in the diagnonal form of Cauchy, we obtain:

$$\left(1+\frac{2}{\mu}\right)^{-\frac{m+2}{2}}\left(1+\frac{1}{\mu}\right)^{-\frac{m+2}{2}} = \sum_{r=0}^{\infty} \frac{\varphi_r(m)}{\mu^{r}} \sum_{s=0}^{\infty} \frac{\psi_s(m)}{\mu^{s}} = \sum_{n=0}^{\infty} \frac{\lambda_n(m)}{\mu^{n}}. \tag{4.25}$$

$$\alpha(\mu, m) = 1 + \frac{1}{\mu}\left\{\binom{m+2}{1} \text{ of members of the form } \zeta_1(m)\right\} + \dots +$$

$$+ \frac{1}{\mu^{s}}\left\{\binom{m+2}{s} \text{ of members of the form } \zeta_s(m)\right\} + \dots +$$

$$+ \frac{1}{\mu^{m+2}}\left\{\binom{m+2}{m+2} \text{ of members of the form } \zeta_{m+2}(m)\right\}.$$

This means

$$\varepsilon(p, m) = \sum_{n=0}^{m+2} \frac{Q_{2n}(m)}{p^n}. \tag{4.26}$$

Extracted in (4.25) and (4.26) the functions, of m are polynomials the powers of which do not exceed the lower index. From (4.23)-(4.26) follows (4.22). Let us note the formula

$$\varepsilon(p, m) = 1 + \frac{m^2+5m+6}{2}\cdot\frac{1}{p} + \frac{3m^4+26m^3+81m^2+106m+48}{24}\times$$
$$\times\frac{1}{p^2} + \ldots + \frac{(m+2)!}{p^{m+2}}. \tag{4.27}$$

Let us place (4.22) into the basic series (3.9) and change there the order of summation. We have

$$g_p(y) = \gamma\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{\infty} P_{2n}(m)\, a_m y^m\right\}\frac{1}{p^n} \tag{4.28}$$

or [see (4.21]

$$\xi(n, y) = \sum_{m=0}^{\infty} P_{2n}(m)\, a_m y^m. \tag{4.29}$$

Let us call that in (4.28) and (4.29) $P_{2n}(m)$ is a polynomial whose power does not exceed 2n.

Let us turn to summation of expressions (4.29), using results of paragraph 2 of this section.

5. According to (3.9), (3.13) and (4.28)

$$g_p(y) = \gamma\left\{\sum_{m=0}^{\infty} a_m y^m + \frac{1}{p}\sum_{m=0}^{\infty}\frac{m^2+3m}{2}\, a_m y^m + \right.$$
$$\left. + \frac{1}{p^2}\sum_{m=0}^{\infty}\frac{3m^4+8m^3-13m^2-44m}{24}\, a_m y^m + \sum_{n=3}^{\infty}\frac{\xi(n, y)}{p^n}\right\}, \tag{4.30}$$

where $\xi(n, y)$ is given by formula (4.29). Since in § 5 concrete calculations will be made for K(y) from (5.1), then it is convenient to assume t = 2 (see p. 2). Using formula (4.5)-(4.7), (4.14) and (4.15) we find:

$$\sum_{m=0}^{\infty} a_m y^m = K(y). \tag{4.31}$$

$$\sum_{m=0}^{\infty} \frac{m^2+2m}{2} a_m y^m = \frac{1}{2} K^{*}(y) - \frac{1}{2} \tilde{K}(y). \tag{4.32}$$

$$\sum_{m=0}^{\infty} \frac{3m^4+8m^3-18m^2-44m}{24} a_m y^m = \frac{1}{8} K^{*}_{(2)}(y) - \frac{1}{24} K^{*}_{(1)}(y) - \\ - \frac{7}{8} K^{*}(y) + \frac{7}{8} \tilde{K}(y). \tag{4.33}$$

since (for instance, according to the Horner method)

$$m^2+2m = (m+2)(m+1) - (m+2).$$
$$3m^4+8m^3-18m^2-44m = (m+2)(m+1)(3m^2-m-21) + 21(m+2).$$

Substituting (4.31)-(4.33) into (4.30) we get

$$g_\mu(y) = \gamma\Big\{K(y) + \frac{1}{\mu}\Big[\frac{1}{2}K^{*}(y) - \frac{1}{2}\tilde{K}(y)\Big] + \\ + \frac{1}{\mu^2}\Big[\frac{1}{8}K^{*}_{(2)}(y) - \frac{1}{24}K^{*}_{(1)}(y) - \frac{7}{8}K^{*}(y) + \frac{7}{8}\tilde{K}(y)\Big] + \sum_{n=3}^{\infty}\frac{\xi(n, y)}{\mu^n}\Big\}. \tag{4.34}$$

Thus for the function

$$g_\mu(y) = \int_0^{\infty} K(ya)\, m_\mu(a)\, da, \tag{4.35}$$

where $m_\mu(a)$ is assigned by relations (2.8) and (2.7) and the nucleus K(ya) is any analytic function, there takes place expansion of (4.34), absolutely convergent in any case to domain (4.20). Quantities entering into expansion of (4.34) are determined by formulas (2.10), (4.6), (4.13), (4.29) and (4.31).

Let us emphasize that results of paragraphs 3 and 4 are accurate for any function K(б) satisfying the conditions formulated in the beginning of § 3.

## § 5. Transparency of Almost Monodispersed System with Fixed Mean-Square Radius

Let us assume that K(δ) sets by formula (1.2). We have

$$K(y)=1-\frac{\sin 2y}{y}+\frac{1-\cos 2y}{2y^2}, \quad K(0)=0. \tag{5.1}$$

Hence

$$K(y)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(4n+2)\,2^{2n}y^{2n}}{(2n+2)!}. \tag{5.2}$$

This means

$$K^{(m)}(0)=\begin{cases}0, & \text{if } m=2r+1\\ (-1)^{r+1}\dfrac{2^{2r}}{r+1}, & \text{if } m=2r,\end{cases} \tag{5.3}$$

Considering (4.18), we obtain

$$\overline{\lim_{m\to\infty}}\sqrt[m]{|K^{(m)}(0)|}=2. \tag{5.4}$$

According to (4.20) and (5.4) we conclude that the basic series (3.9) with K(y) from (5.1) absolutely converges in any case with

$$|y|<\frac{\sqrt{(\mu+2)(\mu+1)}}{2}. \tag{5.5}$$

Let us determine now the expansion of transparency $g_\mu(\frac{x}{2})$ (1.11) into series with respect to $\frac{1}{\mu}$ whe: $r_0=\bar{r}_2$. For this purpose we place in series (4.34) the function (5.1). We have

$$\left.\begin{array}{l}\bar{K}(y)=2-2\cos 2y,\quad K^{*}(y)=2-2\cos 2y+4y\sin 2y\\ K_{(m)}(y)=8y\sin 2y+8y^2\cos 2y,\quad K_{(m)}(y)=8y\sin 2y+\\ \qquad +32y^2\cos 2y-16y^3\sin 2y\end{array}\right| \tag{5.6}$$

Considering formulas (4.13)-(4.15), (4.29) and (5.6), by induction we obtain

$$\xi(n, y) = b_{2n} + \sin 2y \sum_{k=0}^{n} b_{2k-1} y^{2k-1} + \cos 2y \sum_{k=0}^{n-1} b_{2k} y^{2k}. \tag{5.7}$$

where some of the coefficients can be equal to zero. This means that

$$g_0(y) = \gamma \left\{ 1 - \frac{\sin 2y}{y} + \frac{1 - \cos 2y}{2y^2} + \frac{2y \sin 2y}{\nu} + \right.$$
$$\left. + \frac{-\frac{17}{6} y \sin 2y + \frac{11}{3} y^2 \cos 2y - 2y^3 \sin 2y}{\nu^2} + \sum_{n=3}^{\infty} \frac{\xi(n, y)}{\nu^n} \right\}. \tag{5.8}$$

where $\gamma$ is given by (2.10). This series absolutely converges in any case in the domain (5.5).

Let us introduce the designations:

$$K_1(y, \nu) = K(y) + \frac{2y \sin 2y}{\nu}, \tag{5.9}$$

$$K_2(y, \nu) = K_1(y, \nu) + \frac{-\frac{17}{6} y \sin 2y + \frac{11}{3} y^2 \cos 2y - 2y^3 \sin 2y}{\nu^2}, \tag{5.10}$$

where K(y) is determined by formula (5.1). Then

$$g_0(y) \simeq \gamma K(y), \quad g_0(y) \simeq \gamma K_1(y, \nu), \quad g_0(y) \simeq \gamma K_2(y, \nu) \tag{5.11}$$

give a zero, first and second approximation respectively for transparency at large $\nu$, and these formulas have meaning in the region (5.5).

In conclusion let us give the formula for the calculation of transparency (2.9), accurate at $x = 2y$ (1.9) and $\nu$, in the scale (y) accepted here. We have:

$$g_\mu(y)=\gamma\vartheta_\mu(y),\quad \mathrm{tg}\,\omega=\frac{2y}{\sqrt{(\mu+2)(\mu+1)}},$$
$$\vartheta_\mu(y)=1+\frac{1}{2y^2}-\frac{\cos^{\mu+1}\omega}{y}\left[\sqrt{\frac{\mu+1}{\mu+2}}\cos\omega\sin((\mu+2)\omega)+\right.$$
$$\left.+\frac{\cos((\mu+1)\omega)}{2y}\right]. \qquad (5.12)$$

Obviously the region of convergence of series (5.8) corresponds to $|\omega| < \frac{\pi}{4}$.

## § 6. Transparency of Polydispersional Systems for Distributions with a Fixed Mode

Let us take as unity of scale the mode of distribution of particles by dimensions. Then in formulas §§ 1 and 2 one should assume

$$r_0=r_M,\quad \Delta=\mu. \qquad (6.1)$$

Relations (2.5) and (2.6) give

$$g_\mu\left(\frac{x}{2}\right)=\gamma'\left\{1+\frac{3}{\mu}+\frac{2}{\mu^2}+\frac{2}{x^2}-\right.$$
$$\left.-2\frac{\cos^{\mu+1}\omega}{x}\left[\left(1+\frac{1}{\mu}\right)\cos\omega\sin((\mu+2)\omega)+\frac{\cos((\mu+1)\omega)}{x}\right]\right\}. \qquad (6.2)$$

where

$$\gamma'=2\pi N r_M^3,\quad \mathrm{tg}\,\omega=\frac{x}{\mu}. \qquad (6.3)$$

A comparison of formulas (2.9) and (6.2) graphically illustrates the considerations expressed in § 1 with respect to the selection as a linear scale of the quantity the mean-square radius $\bar{r}_2$. Indeed the right side of (2.9) at $x \to \infty$ tends to the general asymptote, the constant number $\gamma$ not dependent on $\mu$. This circumstance does not take place for the right side of (6.2). Moreover, it is accurately only when $r_0 = \bar{r}_2$, which directly follows from the general formula (2.5).

The case $\mu \gg 1$ (the system is almost monodispersed) when $r_0 = r_M$ is exhausted analogous to case $r_0 = \bar{r}_2$ (see § 3-5). We have

$$g_\mu(y) = \gamma' \sum_{m=0}^{\infty} a(\mu, m) a_m y^m, \qquad (6.4)$$

where $\gamma'$ is given by (6.3), $a_m$ by (3.8), and $a(\mu, m)$ by (4.24). The radius of convergence of power series (6.4) is determined by the inequality

$$R \geqslant \frac{\mu}{\overline{\lim\limits_{m\to\infty}} \sqrt[m]{|K^{(m)}(0)|}}. \qquad (6.5)$$

According to (4.26) and (4.27), formula (6.4) can be rewritten in the form

$$g_\mu(y) = \gamma' \Big\{ K(y) + \frac{1}{\mu}\Big[\frac{1}{2} K^{*}(y) + \tilde{K}(y)\Big] + \\ + \frac{1}{\mu^2}\Big[\frac{1}{8} K^{*}_{(2)}(y) + \frac{17}{24} K^{*}_{(1)}(y) + K^{*}(y)\Big] + \sum_{n=3}^{\infty} \frac{\eta(n, y)}{\mu^n} \Big\}. \qquad (6.6)$$

Here designations (4.6), (4.13), (6.3) are used, and

$$\eta(n, y) = \sum_{m=n-2}^{\infty} Q_{2n}(m) a_m y^m, \qquad (6.7)$$

where $Q_{2n}(m)$ is a polynomial whose degree does not exceed $2n$. Let us assume now that $K(y)$ is assigned by formula (5.1). According to (5.4) and (5.6), in this case we obtain

$$g_\mu(y) = \gamma' \Big\{ 1 - \frac{\sin 2y}{y} + \frac{1-\cos 2y}{2y^2} + \frac{3 - 3\cos 2y + 2y \sin 2y}{\mu} + \\ + \frac{2 - 2\cos 2y + \frac{32}{3} y \sin 2y + \frac{29}{3} y^2 \cos 2y - 2y^3 \sin 2y}{\mu^2} + \sum_{n=3}^{\infty} \frac{\eta(n, y)}{\mu^n} \Big\}, \qquad (6.8)$$

where the indicated expansion is accurate in any case if

$$|y| < \frac{\mu}{2}. \qquad (6.9)$$

and $\eta(n, y)$ has the form

$$\eta(n, y) = \sin 2y \sum_{k=0}^{n} b_{2k-1} y^{2k-1} + \cos 2y \sum_{k=0}^{n-1} b_{2k} y^{2k} + P_{n-3}(y) \quad (n > 3), \tag{6.10}$$

where $P_{n-3}(y)$ is a polynomial whose degree is not higher than $n - 3$.

## § 7. The Bond between Dimensionless Transparencies at Different Linear Scales. The case of $\mu$ Integers

As the model of the $\delta$-shaped sequence we will, as above, use totality of gammadistributions with different $\mu$ (see § 1):

$$f_\mu^*(r) = N \frac{\mu^{\mu+1}}{\Gamma(\mu+1)} r^\mu e^{-\mu r} \text{ when } f_\mu(a) = N r_0^3 \frac{\Delta^{\mu+1}}{\Gamma(\mu+1)} a^\mu e^{-\Delta a}, \tag{7.1}$$

where

$$r = a r_0, \quad f_\mu(a) = r_0^4 f_\mu^*(r), \quad \Delta = \beta r_0. \tag{7.2}$$

Parameter $\mu$ characterizes the width of the distribution (7.1) [for instance, by formula (1.7), i.e.,] the degreee of monodispersiveness of the system; at $\mu \to \infty$ the system becomes monodispersed. In turn the parameter $\beta$ at fixed width of distribution and selected type of the scale $r_0$($r_0 = r_M$, $r_0 = \bar{r}$, etc.) is determined by the magnitude of this scale. Tnus (in § 7 all parameters dependent on $\mu$ will be provided by a subscript, for instance: $\Delta = \Delta_\mu$),

$$\Delta_\mu = \begin{cases} \mu, & \text{if } r_0 = r_M \\ \mu + 1, & \text{if } r_0 = \bar{r} \\ \sqrt{(\mu+2)(\mu+1)}, & \text{if } r_0 = \bar{r}_2 \end{cases} \tag{7.3}$$

and, correspondingly,

$$\beta_\mu = \frac{\Delta_\mu}{r_0}. \tag{7.4}$$

Let us assume that $\beta_\mu$ is secured. Then for any fixed $\mu$ we have

$$\frac{\bar{r}_2}{r_0} = \frac{\tau_\mu}{\Lambda_\mu}, \text{where } \tau_\mu = \sqrt{(\mu+2)(\mu+1)}. \tag{7.5}$$

In (7.3) there are extracted values of $\Lambda_\mu$ for cases $r_0 = r_M$, $\bar{r}$ and $\bar{r}_2$. Let us designate (see § 1):

$$y = \beta\nu^* \bar{r}_2, \quad \bar{y} = \beta\nu^* r_0, \quad \beta = 2\pi(m-1); \tag{7.6}$$

$$\gamma = 2\pi N \bar{r}_2^2, \quad \gamma_0 = 2\pi N r_0^2. \tag{7.7}$$

According to (7.5) and (7.6),

$$\frac{y}{\bar{y}} = \frac{\tau_\mu}{\Lambda_\mu}. \tag{7.8}$$

Quantities $\beta\nu^*$ and $g_\mu^*(\nu^*)$ are invariant with respect to the type of selected scale $r_0$. We have:

$$\beta\nu^* = \frac{y}{\bar{r}_2} = \frac{\bar{y}}{r_0}, \tag{7.9}$$

$$g_\mu^*(\nu^*) = \frac{g_\mu(y)}{\bar{r}_2} = \frac{\bar{g}_\mu(\bar{y})}{r_0}. \tag{7.10}$$

Let us introduce the dimensionless characteristic of the transparency

$$G_\mu = \frac{g_\mu^*(\nu^*)}{2\pi N \bar{r}_2^2}. \tag{7.11}$$

The presence in the denominator $\bar{r}_2$ and not any other linear scale $r_0 \neq r_2$ is explained in that only in this case (see § 1 and 5) there has place

$$\lim_{\mu\to\infty} G_\mu = 1 \quad (\mu \text{ – random}). \tag{7.12}$$

Formally this relation ensues from (5.12), (7.7), (7.10) and (7.11). Really,

$$Q_\mu = \frac{g_\mu(y)}{\gamma} = \varphi_\mu(y) \to 1 \text{ when } y \to \infty. \tag{7.13}$$

Let us establish now the bond between dimensionless transparencies $g_\mu(y)$ and $\tilde{g}_\mu(\tilde{y})$ from (7.10). Let us assume that at an arbitrary linear scale of $r_0$ the following formula is accurate

$$\tilde{g}_\mu(\tilde{y}) = \gamma_0 \Psi_\mu \varphi_\mu\left(\frac{2\tilde{y}}{\Delta_\mu}\right). \tag{7.14}$$

where $\Psi_\mu$ will be selected from the condition

$$\lim_{x\to\infty} \varphi_\mu(x) = 1. \tag{7.15}$$

Then on the basis of (7.5), (7.7), (7.8), (7.10) and (7.14), we obtain

$$g_\mu(y) = \gamma \frac{\Psi_\mu \Delta_\mu^2}{\tau_\mu^2} \varphi_\mu\left(\frac{2y}{\tau_\mu}\right). \tag{7.16}$$

Parameter $\Psi_\mu$ can be determined in terms of $\Delta_\mu$ and $\tau_\mu$. Indeed, in virtue of (7.13) and (7.15),

$$\lim_{y\to\infty} \frac{g_\mu(y)}{\gamma} = \lim_{y\to\infty} \varphi_\mu\left(\frac{2y}{\tau_\mu}\right) = 1, \tag{7.17}$$

and therefore

$$\frac{\Psi_\mu \Delta_\mu^2}{\tau_\mu^2} = 1 \tag{7.18}$$

or

$$\Psi_\mu = \frac{(\mu+3)(\mu+1)}{\Delta_\mu^2} \quad (r_0 \text{ and } \mu \text{ - random}). \tag{7.19}$$

Let us sum up. If for arbitrary $r_0$

$$\bar{g}_\mu(\bar{y}) = \tau_0 \Psi_\mu \theta_\mu\left(\frac{2\bar{y}}{\Delta_\mu}\right), \quad \lim_{x\to\infty} \theta_\mu(x) = 1, \tag{7.20}$$

then $\Psi_\mu$ is determined by (7.19),

$$g_\mu(y) = \tau \theta_\mu\left(\frac{2y}{\tau_\mu}\right) \tag{7.21}$$

and considering (7.13),

$$Q_\mu = \Phi_\mu(y) = \Phi_\mu\left(\frac{2y}{\tau_\mu}\right). \tag{7.22}$$

where $\Phi_\mu(y)$ is given by (5.12), and $\tau_\mu$ by (7.5).

A comparison of formulas for the transparency of (7.20) and (7.21) shows how with a change in the type of linear scale $r_0$ the variables are expanded (form of dependency, of course, is kept). Thus with transition from $r_0$ to $\bar{r}_2$ in virtue of (7.20) and (7.21) or (7.8) and (7.10):

$$\left.\begin{aligned} \bar{y} &= \frac{\Delta_\mu}{\tau_\mu} y \\ \bar{g}_\mu &= \frac{\Delta_\mu}{\tau_\mu} g_\mu \end{aligned}\right| \tag{7.23}$$

In [5] the formula is obtained for calculation of the transparency of systems with gammadistribution with $\mu$ integers in the case $r_0 = r_M$:

$$\left.\begin{aligned} &\bar{g}_\mu(\bar{y}) = 2\pi N r_M^2 \left(1+\frac{1}{\mu}\right)\left(1+\frac{2}{\mu}\right)\Phi_\mu\left(\frac{2\bar{y}}{\mu}\right) \quad (\mu>0) \\ &\Phi_\mu(x) = 1 + \\ &+ \frac{2}{\tau_\mu}\left[x^{-2} - \frac{x^{-2} + \frac{(\mu+3)(\mu+2)}{2}\sum\limits_{m=0}^{\left[\frac{\mu+1}{2}\right]}(-1)^m\binom{\mu+1}{2m}\frac{x^{2m}}{2m+1}}{(1+x^2)^{\mu+2}}\right] \end{aligned}\right\} \tag{7.24}$$

This means that in accordance with (7.3) and (7.19) we have

$$\Delta_\mu = \mu, \quad \nabla_\mu = \frac{(\mu+2)(\mu+1)}{\mu^2} \quad (\mu > 0, \quad r_0 = r_M) \tag{7.25}$$

and according to the transition (7.20)-(7.21), we obtain that at $\mu$ integers

$$g_\mu(y) = \gamma\theta_\mu\left(\frac{2y}{\sqrt{(\mu+2)(\mu+1)}}\right), \tag{7.26}$$

where $\gamma$ is given by (7.7), and $\theta_\mu(x)$ by (7.24).

In [5] there is also obtained the formula for transparency when $\mu = 0$

$$\bar{g}_0(\bar{y}) = 2\pi N \bar{r}^2 2\theta_0(2\bar{y}), \quad \theta_0(x) = \frac{x^2(3+x^2)}{(1+x^2)^3}. \tag{7.27}$$

This means that in accordance with (7.3) and (7.19) we have:

$$\Delta_0 = 1, \quad \nabla_0 = 2 \quad (\mu = 0, \quad r_0 = \bar{r}) \tag{7.28}$$

and, according to the transition (7.20)-(7.21), we arrive at the conclusion that formula (7.26) is accurate at $\mu = 0$ if $\theta_0(x)$ is taken from (7.27).

Considering (7.3) and (7.8) we find

$$y = \frac{\sqrt{(\mu+2)(\mu+1)}}{\mu}\bar{y} \quad (\mu > 0, \quad r_0 = r_M),$$
$$y = \sqrt{2}\,\bar{y} \quad (\mu = 0; \quad r_0 = \bar{r}). \tag{7.29}$$

Thus when $\mu \geq 0$ (integers) the following formula is accurate

$$g_\mu^*(v^*) = 2\pi N \bar{r}^2 \theta_\mu\left(\frac{2\mu^* \bar{r}_*}{\sqrt{(\mu+2)(\mu+1)}}\right), \tag{7.30}$$

where $\theta_\mu(x)$ is calculated with respect to (7.24) if $\mu > 0$ and with respect to (7.27) if $\mu = 0$.

Formulas (7.26) and (7.30) are convenient during calculation of the transparency for small integers $\mu \geq 0$ (up to $\mu \sim 10$ to 12).

Table 1

| $\mu$ | $\bar{y}_{M_\mu}$ | $y_{M_\mu}$ |
|---|---|---|
| 0 | 0,866 | 1,22 |
| 2 | 0,870 | 1,51 |
| 6 | 1,35 | 1,68 |
| 10 | 1,56 | 1,78 |

Examples. In §§ 10 and 11 from [6] there are extracted values of $x_{M_\mu}$, in which the transparency $g_\mu(\frac{x}{2})$ attains a maximum value. There for $\mu = 0$ $r_0 = \bar{r}$ was accepted, and for $\mu$, equal to 2, 6, and 10, $r_0 = r_M$. Assuming $x = 2\tilde{y}$ [see (1.9)] we find values

$$\bar{y}_{M_\mu} = \frac{x_{M_\mu}}{2}, \tag{7.31}$$

in which $g_\mu(y)$ is maximum (Table 1). Let us note that as follows from Table 1 [6], the transparency $g^*_\mu(\nu^*)$ with $\mu$ equal to 0, 2, and 6 has a single maximum. When $\mu = 10$ there exists still a second maximum of transparency (see Table 4). This maximum is considerably weaker and is not considered here.

According to (7.29), $y = y_{M_\mu}$, in which $g_\mu(y)$ is maximum, are found by formulas:

$$y_{M_\mu} = \frac{\sqrt{(\mu+2)(\mu+1)}}{\mu}\bar{y}_{M_\mu} \quad (\mu > 0),$$
$$y_{M_0} = \sqrt{2}\,\bar{y}_{M_0} \quad (\mu = 0). \tag{7.32}$$

In Table 1 there are given v a l u e s $\tilde{y}_{M_\mu}$ and $y_{M_\mu}$ with $\mu$ equal to 0, 2, 6, and 10.

Thus we converted to the general linear scale $r_0 = \bar{r}_2$, accepted for variable y [see (7.6)]. Let us note that on the basis (7.3) and (7.4),

$$r_M < \bar{r} < \bar{r}_2 \quad (\beta_\mu \text{ is fixed}). \tag{7.33}$$

Therefore, in virtue of (7.9) these inequrlities are accurate

$$\bar{y}_{r_0=r_M} < \bar{y}_{r_0=\bar{r}} < y_{r_0=\bar{r}_2}. \tag{7.34}$$

Here there are extracted values of y at various scales of $r_0$, corresponding to the same wave number $\nu^*$.

§ 8. Spectral Transmittance of Polydispersional Systems with Different Width of Distribution

For an illustration of obtained formulas there were calculated curves of spectral transmittance of different polydispersional systems (Figs. 1-6). The curves are plotted in dimensionless variables. Along the axis of the abscissas is plotted the magnitude

$$y = \beta\nu^{*}\bar{r}_{2} = (m-1)\,\bar{\rho}_{2}, \tag{8.1}$$

where

$$\bar{\rho}_{2} = \frac{2\pi\bar{r}_{2}}{\lambda}, \quad \bar{r}_{2} = \sqrt{\bar{r}^{2}}, \tag{8.2}$$

and along the axis of the ordinates

$$G_{\mu} = \frac{\varepsilon_{\mu}^{*}(\nu^{*})}{2\pi N\bar{r}^{2}}. \tag{8.3}$$

On every figure there is given few curves of $G_{\mu}$, where designations of curves I-IV on all figures have identical meaning. These curves are calculated by the formulas (5.1), (5.9), (5.10) and (5.12) respectively. Curves I, II, III signify, correspondingly the zero, first and second approximation of transparency at large $\mu$ (see § 5). Curve IV describes the exact movement of the transparency of the system.

In Table 2, proceeding from calculations of the curves depicted in Figs. 1-6, for the set $\mu$,

$$\mu = 10,\ 20,\ 30,\ 50,\ 100 \text{ and } 400 \tag{8.4}$$

there are estimated the quantities $y_0$, $y_1$ and $y_2$ determining the limiting values of $y$ for the corresponding approximation (at maximum permissible absolute deviation from the exact curve not exceeding 10%). The relative width of the considered distributions $\Delta\varepsilon$ (1.7) is given in Table 3.

Let us note that Table 2 does not completely reflect the degree of proximity of considered approximations to the exact transparency.

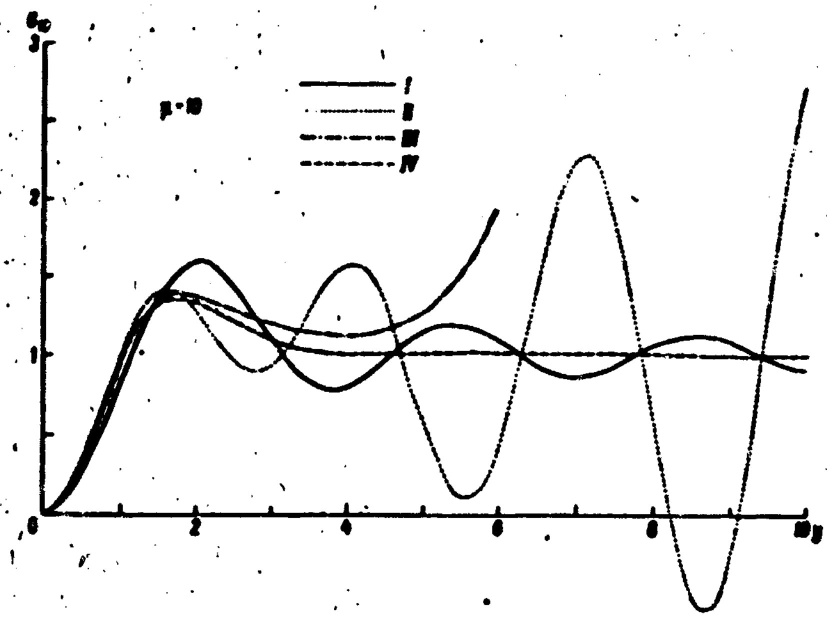

Fig. 1.

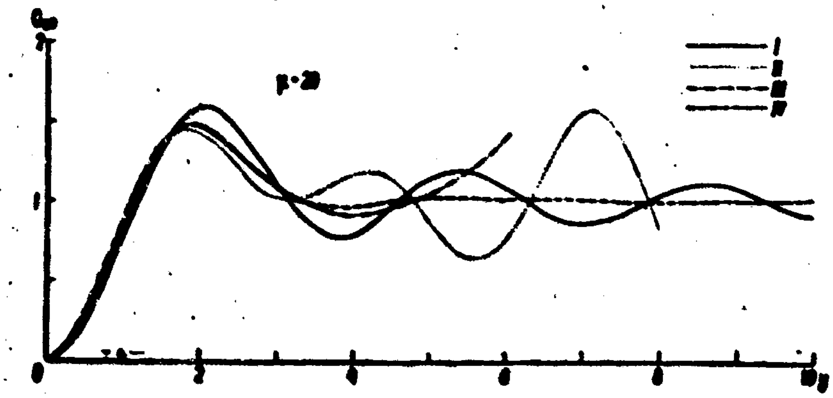

Fig. 2.

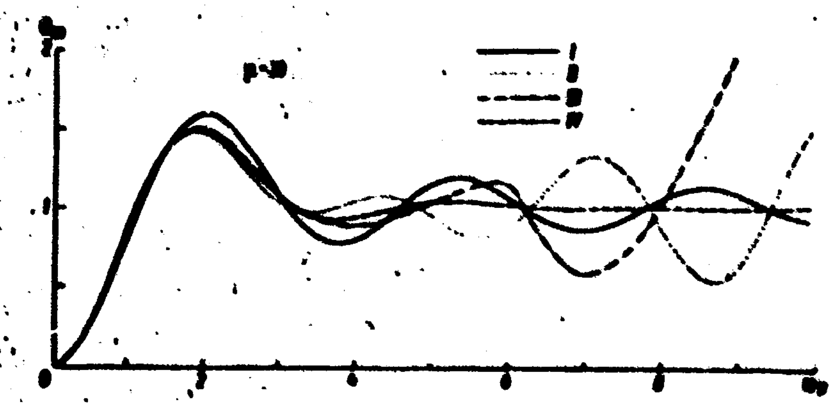

Fig. 3.

In virtue of the selection of the scale $r_0 = \bar{r}_2$, transparencies $\Phi_\mu(y)$ at all $\mu$ (including when $\mu = \infty$) have a general asymptote when $y \to \infty$. Therefore, zero approximation $K(y) = G_\infty = \Phi_\infty(y)$ is naturally, close to $G_\mu = \Phi_\mu(y)$ at large y. If $K(y)$ at $y \lesssim 8$ to 10 differs from $\Phi_\mu(y)$ by not more than 10% (this takes place from the general considerations for sufficiently large $\mu$), then at all other y good coincidence of $\Phi_\mu(y)$ and $K(y)$ is ensured. Therefore, for instance, in Table 2 $y_0 > y_1$ with $\mu$, equal to 100.

Table 2

| $\mu$ | $y_0$ | $y_1$ | $y_2$ |
|---|---|---|---|
| 10 | 1.7 | 2.2 | 3.9 |
| [illegible] | 2.4 | 3.6 | 5.3 |
| [illegible] | 3.3 | 5.1 | 6.4 |
| [illegible] | 4.0 | 6.7 | 6.9 |
| 100 | >15.0 | 11.4 | >15.0 |
| 400 | >15.0 | >15.0 | >15.0 |

At the same time the first and second approximations considerably better than the zero describe the transparency $\Phi_\mu(y)$ on the most important section of the change in frequencies corresponding to small $y$ ($0 \leq y \gtrsim 8$ to 10) (see Figs 1-6).

Table 3

| $\mu$ | $k$ | $y_{M_\mu}$ | $G_{M_\mu}$ | $y_{m_\mu}$ | $G_{m_\mu}$ |
|---|---|---|---|---|---|
| 0 | $\infty$ | 1.22 | 1.12 | — | — |
| 2 | 1.754 | 1.51 | 1.22 | — | — |
| 6 | 1.013 | 1.[illegible] | 1.33 | — | — |
| 10 | 0.784 | 1.7[illegible] | 1.38 | — | — |
| [illegible] | 0.555 | 1.[illegible] | 1.46 | 3.71 | 0.95 |
| [illegible] | 0.453 | 1.[illegible] | 1.50 | 3.73 | 0.92 |
| [illegible] | 0.351 | 1.95 | 1.52 | 3.75 | 0.88 |
| 100 | 0.248 | 2.00 | 1.55 | 3.76 | 0.83 |
| 400 | 0.124 | 2.04 | 1.57 | 3.79 | 0.79 |
| $\infty$ | 0 | 2.05 | 1.59 | 3.80 | 0.77 |

It is necessary to note that the second approximation does not give considerable improvement of the result as compared to the first, and at $\mu \gtrsim 100$ the first approximation practically quite accurately presents the transparency for all considered range of values of y. Therefore, curve III in Figs. 5 and 6 is not depicted.

In Table 4 for the set $\mu$ (8.4) accurate values of transparencies are calculated [see (7.22)]

$$G_\mu = \Phi_\mu(y). \tag{8.5}$$

where $\Phi_\mu(y)$ is given by (5.12). As follows from results of § 7 [transition (7.20)-(7.21)] the following formula is accurate

$$\Phi_\mu(y) = \Phi_0\left(\frac{2y}{\sqrt{(\mu+2)(\mu+1)}}\right). \tag{8.6}$$

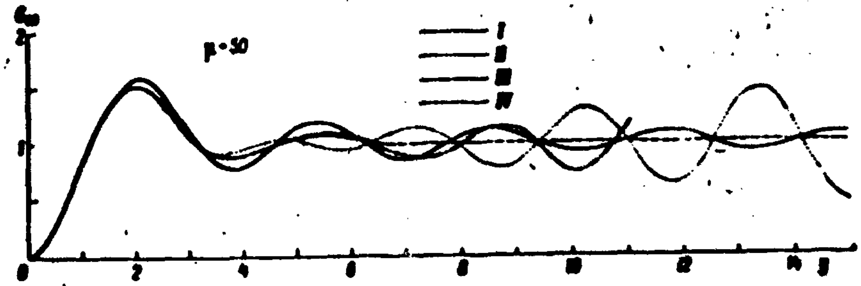

Fig. 4.

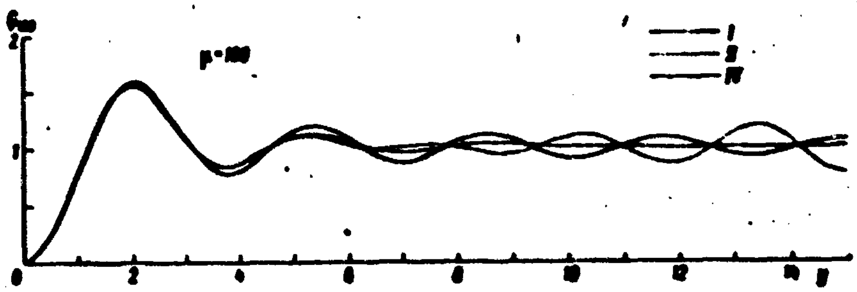

Fig. 5.

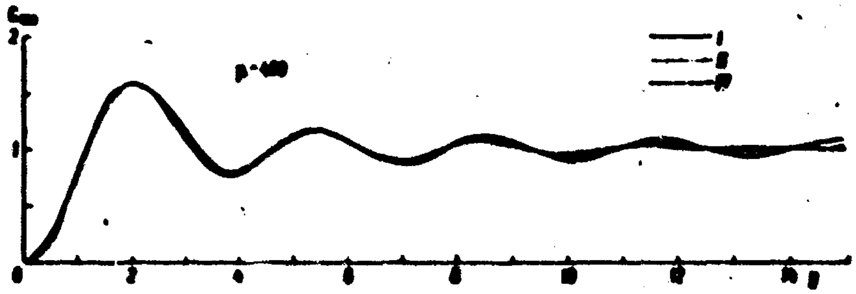

Fig. 6.

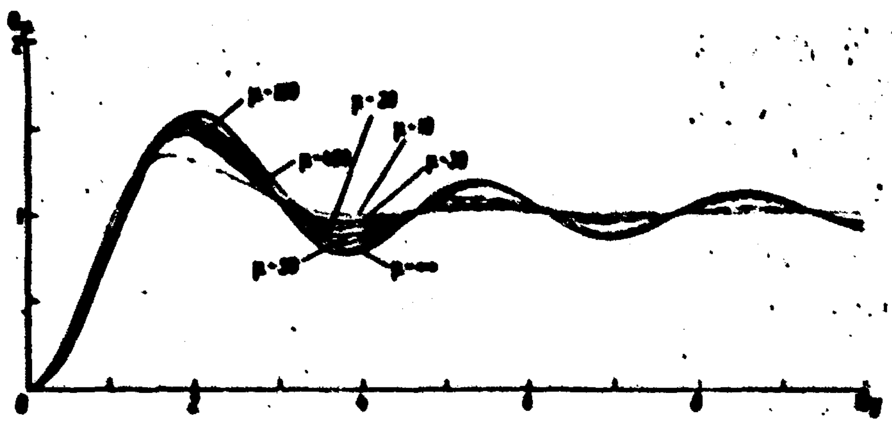

Fig. 7. Accurate transparency for the set μ.

allowing to pass from the transparency obtained with the arbitrary type of the linear scale $r_0$, to the transparency in the scale $r_0 = \bar{r}_2$ and inversely. An example in this case can be the transition with integers $\mu \geq 0$ from (7.24) and (7.27) to (7.26).

Table 4.

| $y$ | $\theta_{10}(y)$ | $\theta_{20}(y)$ | $\theta_{30}(y)$ | $y$ | $\theta_{50}(y)$ | $\theta_{100}(y)$ | $\theta_{400}(y)$ |
|---|---|---|---|---|---|---|---|
| 0,5 | 0,309 | 0,278 | 0,249 | 0,5 | 0,258 | 0,234 | 0,184 |
| 1,0 | 0,933 | 0,868 | 0,852 | 1,0 | 0,830 | 0,768 | 0,806 |
| 1,5 | 1,334 | 1,351 | 1,363 | 1,5 | 1,356 | 1,327 | 1,413 |
| 2,0 | 1,316 | 1,455 | 1,495 | 2,0 | 1,530 | 1,558 | 1,581 |
| 2,5 | 1,218 | 1,283 | 1,318 | 2,5 | 1,359 | 1,388 | 1,458 |
| 3,0 | 1,084 | 1,076 | 1,076 | 3,0 | 1,076 | 1,070 | 1,165 |
| 3,5 | 1,022 | 0,972 | 0,940 | 3,5 | 0,898 | 0,864 | 0,899 |
| 4,0 | 1,009 | 0,970 | 0,940 | 4,0 | 0,901 | 0,858 | 0,810 |
| 4,5 | 1,012 | 1,003 | 0,999 | 4,5 | 0,995 | 0,986 | 0,945 |
| 5,0 | 1,015 | 1,026 | 1,042 | 5,0 | 1,066 | 1,102 | 1,105 |
| 5,5 | 1,016 | 1,029 | 1,045 | 5,5 | 1,074 | 1,116 | 1,179 |
| 6,0 | 1,015 | 1,022 | 1,028 | 6,0 | 1,056 | 1,050 | 1,076 |
| 6,5 | 1,013 | 1,014 | 1,012 | 6,5 | 1,001 | 0,983 | 0,956 |
| 7,0 | 1,011 | 1,010 | 1,004 | 7,0 | 0,988 | 0,956 | 0,894 |
| 7,5 | 1,009 | 1,008 | 1,004 | 7,5 | 0,996 | 0,981 | 0,962 |
| 8,0 | 1,008 | 1,007 | 1,006 | 8,0 | 1,008 | 1,020 | 1,044 |
| 8,5 | 1,007 | 1,007 | 1,007 | 8,5 | 1,013 | 1,035 | 1,066 |
| 9,0 | 1,006 | 1,006 | 1,007 | 9,0 | 1,011 | 1,027 | 1,069 |
| 9,5 | 1,006 | 1,006 | 1,006 | 9,5 | 1,007 | 1,003 | 0,981 |
| 10,0 | 1,005 | 1,005 | 1,005 | 10,0 | 1,004 | 1,002 | 0,950 |

| $y$ | $\theta_{50}(y)$ | $\theta_{100}(y)$ | $\theta_{400}(y)$ |
|---|---|---|---|
| 10,5 | 1,003 | 0,995 | 0,960 |
| 11,0 | 1,003 | 1,003 | 1,000 |
| 11,5 | 1,004 | 1,008 | 1,043 |
| 12,0 | 1,004 | 1,008 | 1,037 |
| 12,5 | 1,004 | 1,005 | 1,006 |
| 13,0 | 1,003 | 1,002 | 0,979 |
| 13,5 | 1,003 | 1,000 | 0,978 |
| 14,0 | 1,002 | 1,003 | 0,999 |
| 14,5 | 1,002 | 1,003 | 1,020 |
| 15,0 | 1,002 | 1,003 | 1,024 |

Figure 7 gives curves of exact transparencies for the set $\mu$ (8.4) and $\mu = \infty$ [monodispersed case, see (8.9)]. From the figure one can see how with the growth of $\mu$ and corresponding narrowing of the width of distribution $\Delta\varepsilon$ components of the spectral structure become more distinct, there appears even more waves on the curve of transparency.

In the examples considered on the conversion of spectral transmittance, [6] for a description of the right segment of curve $G_\mu$ we used the assumption on the monotonous decrease in $g^*(\nu^*)$ for the first maximum. From Fig. 7 one can see that when $\mu < 20$, in virtue of the stability of the calculation scheme, it is practically possible to consider that this takes place. Data of aerologic investigations show that distributions of radii of particles of natural clouds and fogs are usually described when $\mu < 20$. This means that examples of the analysis of the transparency given in [6] are typical.

At the same time the calculation scheme from [6] permits not only determining the spectrum of particles in the case of multimodal transparency but also using there appendices of the function $\omega(y)$, $\omega_0(y)$ and $\omega_2(y)$ in Table 1. The tables shown are calculated for the scale

$$r_0 = \frac{2}{\beta\tau^*}, \qquad (8.7)$$

where $\tau^*$ is the range of the change in wave numbers $\nu^*$. If $\tau^*$ is great, then $r_0$ is small, and it is necessary to take large a in the formula

$$r = ar_0, \qquad (8.8)$$

in order to calculate the spectrum of particles $f^*(r)$ in the needed interval of dimensions. This means that for large $\tau^*$ of Table 1 the appendices from [6], possibly, must be continued for the large interval of values a as compared to the interval considered there.

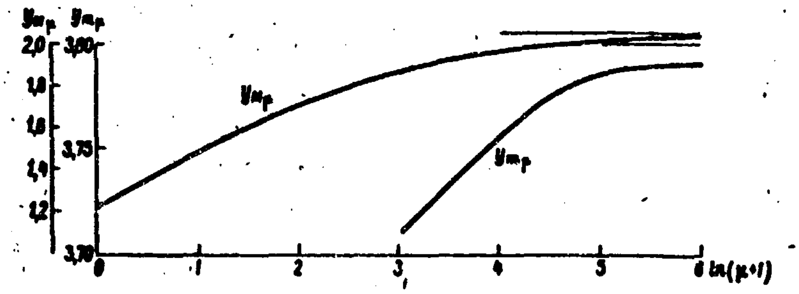

Fig. 8. Abscissas of first maxima and minima of transparencies.

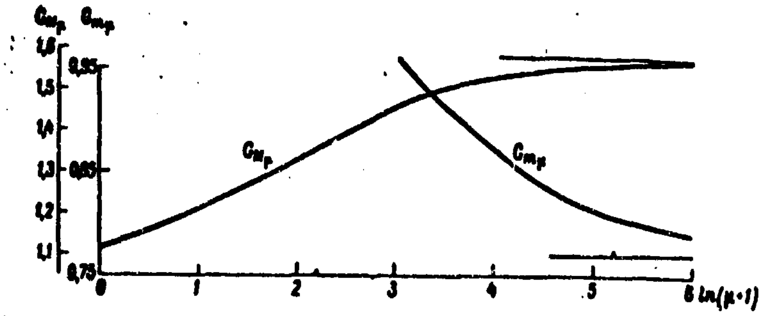

Fig. 9. Ordinates of first maxima and minima of transparencies.

Table 3 gives values $y_{M_\mu}$ and $y_{m_\mu}$ in which $G_\mu$ attains, correspondingly, the first maximum and first minimum (if it exists)

$G_{M_\mu}$ and $G_{m_\mu}$. The limiting case $\mu = \infty(\Delta\varepsilon = 0)$ corresponds to monodispersiveness of the system of particles where obviously,

$$G_\infty = K(y). \tag{8.9}$$

where K(y) is given by (1.2). The other limiting case $\mu = 0(\Delta\varepsilon = \infty)$ corresponds to the widest distribution in the accepted family of models of spectra of particles.

Figures 8 and 9 show, correspondingly the movement of the dependence of $y_{M_\mu}$, $y_{m_\mu}$ and $G_{M_\mu}$, $G_{m_\mu}$ on $\mu$. The scale along the axis of the abscissas in Figs. 8 and 9 is accepted for convenience of the image logarithmic. Let us note that extrema $G_\mu = \vartheta_\mu(y)$ are very smooth, and therefore their reliable determination requires very exact calculations. The clear minimum $\vartheta_\mu(y)$ is observed when $\mu > 10$. The position of abscissas of the first minima $\vartheta_\mu(y)$ is almost not changed with the growth in $\mu$ – all $y_{m_\mu}$ are concentrated in the narrow interval $3.7 \leq y \leq 3.8$.

Figures 8 and 9 illustrate the character of the change of transparency with gradual transition from the polydispersional system of particles to the monodispersed with a decrease in the width of the distribution of the model.

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